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An exact solution of the time-dependent Ginzburg-Landau equations for propagating fronts in superconductor

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Abstract We have found an exact solution of the time-dependent Ginzburg-Landau (TDGL) equations in connection with the problem of front (interface) propagation in superconductor Here, we find a distinct solution of the mentioned problem found earlier by Di Bartolo and Dorsey [*Phys Rev Lett* 77 4442 (1996)] with a front speed c = 2 828 and display some of its properties

Keywords Ginzburg-Landau equation, exact solution, front propagation in superconductor

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1. Introduction

We consider the following problem in a bulk superconductor. There is an applied magnetic field equal to the critical field H_c. This gives rise to a separation of the normal and superconducting phase through the mechanism of a planar superconducting-normal interface. Now reduce the applied field to zero, rapidly. The interface becomes unstable and moves towards the normal phase and expels any trapped magnetic flux, and leaves the bulk sample in the Meissner phase. If one assumes that the interface continuous to be planar, one can enquire about its dynamics. There are many such types of phenomena occurring in physics [see Ref.[1,2] and references therein] which form fronts (interfaces). Selection of the (constant, unique) speed of the front is a common issue to all.

Di Bartolo and Dorsey [2] found an exact solution of the time-dependent Ginzburg-Landau (TDGL) equations for the situation mentioned above and showed among other things, that constant velocity fronts propagate at a unique speed determined by the magnitude of the magnetic flux trapped in the front. They say, " To the best of our knowledge, this is the only known exact solution of the TDGL equation". Here, we find an exact solution for the same problem for a particular set of parameters that is distinct to that found by them. The new solution, in a certain sense, has a 'longer range' than the earlier solution [2], somewhat akin to the difference between a 'Yukawa-type' and 'Coulomb-type' potential. Some properties of the solution and possible connections with other problems are considered.

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2. Field equations and the earlier solution

The dynamic equation related to the problem of front propagation in superconductor is the well-known one-dimensional TDGL equations [1,2]. In dimensionless units, these are

$$\frac{\partial f}{\partial f} = \frac{1}{\kappa^2}\frac{\partial^2 f}{\partial x^2} - q^2 f + f - f^3 \tag{1}$$

$$\bar{\sigma}\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial x^2} - f^2 q, \tag{2}$$

where $f(x, t)$ is related to the magnitude of the superconducting order parameter $\Psi = \left[f(x,t)\exp[i\varphi(x,t)]\right]$, $q(x,t) =$

$A - \nabla\phi(x,t)/\kappa$ is the gauge-invariant vector potential related to the magnetic field $h = \partial q/\partial x$, κ is the Ginzburg-Landau parameter and $\bar{\sigma}$ is the dimensionless normal state conductivity.

For steady travelling wave solutions of eqs.(1) and (2), express $f(x,t)$ and $q(x,t)$ in the form:

$$f(x,t) = F(X),\ q(x,t) = Q(X),$$

where $X = x - ct$, with $c > 0$. (3)

Eqs.(1) and (2) then become (denoting $F' = dF/dX$, $Q' = dQ/dX$, *etc.*)

$$\frac{1}{\kappa^2}F'' + cF' - Q^2F + F - F^3 = 0\,, \tag{4}$$

$$Q'' + \bar{\sigma}cQ' - F^2Q = 0\,. \tag{5}$$

For an exact solution, Di Bartolo and Dorsey make the following *ansatz*

$$F + Q = 1, \tag{6}$$

and find that this is consistent only when $\kappa = 1/\sqrt{2}$, $\bar{\sigma} = 1/2$ in which case, the equation for the order parameter reduces to the following:

$$F'' + \frac{c}{2}F' + F^2 - F^3 = 0\,. \tag{7}$$

They find that at a special point, the order parameter and vector potential are *dual* leading to the exact solution with $c = 1.414$, of the form :

$$F(X) = \frac{1}{\exp\left(X/\sqrt{2}\right)+1},\quad Q(X) = \frac{1}{\exp\left(-X/\sqrt{2}\right)+1}, \tag{8}$$

and discuss various properties and implication of the solution.

For comparison, Di Bartolo and Dorsey considered the similar type of equation (of course, used for different problem), the one-dimensional Fisher's equation [see Ref. [2] and references therein]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u)\,, \tag{9}$$

where $u(> 0)$ is likely to be an order parameter and the function $F(u)$ follows $F(0) = F(1) = 0$. The solution of eq.(9) evolves into fronts of the form $u(x,t) = U(x - ct)$ which connect $u = 1$ at $x = -\infty$ to $u = 0$ at $x = \infty$; the speed of the front satisfy $2\sqrt{F'(0)} \le c \le 2\sup\sqrt{F(u)/u}$, so that for the special case $F(u) = u - u^3$, the selected speed is $c = 2$.

Now, if we consider that the vector potential in eq.(1) is zero, then eq.(1) is exactly the Fisher's equation [eq.(9)], with fronts which propagate at a speed $c = 2/\kappa$ in dimensionless unit

If the relation between $F(X)$ and $Q(X)$ exists at the special point $\kappa = 1/\sqrt{2}$, called the *dual point* [3], the selected speed should be $c = 2.828$ and we achieved this goal without any perturbative calculation.

3. The new solution

The new solution is as follows. We proceed in a straightforward manner and verify that the solution satisfies eqs.(4) and (5) if the parameters satisfy certain relations :

$$F(X) = \frac{\sqrt{2}}{\eta + X}\,,\quad Q(X) = 1 - \frac{c/2}{\eta + X} \tag{10}$$

Here, η is an arbitrary constant. From eq.(10), we get

$$F'(X) = -\frac{\sqrt{2}}{(\eta + X)^2};\quad F''(X) = \frac{2\sqrt{2}}{(\eta + X)^3} \tag{11a}$$

$$Q^2(X) = 1 - \frac{\quad}{(\eta + X)}, - \frac{c^2/4}{(\eta + X)^2}; \tag{11b}$$

$$Q'(X) = \frac{c/2}{(\eta + X)^2};\ Q''(X) = -\frac{c}{(\eta + X)}, \tag{11c}$$

These expressions have been given explicitly so that the reader can readily verify the solution. Substituting from eqs.(11a,b,c), we find that eqs.(4) and (5) are satisfied for arbitrary η, c if we choose κ and $\bar{\sigma}$ in terms of c as follows:

$$\kappa^2 = \frac{8}{\left(c^2 + 8\right)};\quad \sigma = \frac{\quad}{2}\,. \tag{12}$$

Note that $\kappa < 1$, and as c increases, κ and $\bar{\sigma}$ decrease; and that eq. (10) implies

$$\left(c/2\sqrt{2}\right)F + Q = 1\,, \tag{13}$$

which is a more general relation than (6), reducing to the latter only when $c = 2.828$. In any case, the solution (10) is quite distinct from the earlier solution [eq.(8)].

If we introduce [cf. Ref.2] the *outer variable* $\xi = cX$, the eqs.(4) and (5) implies

$$\frac{c^2}{\kappa^2}f''(\xi) + c^2f'(\xi) - q^2(\xi)f(\xi) + f(\xi) - f^3(\xi) = 0\,, \tag{14}$$

$$c^2q''(\xi) + \bar{\sigma}c^2q'(\xi) - f^2(\xi)q(\xi) = 0\,, \tag{15}$$

then the exact solution [eq.(10)] is also satisfied and follows that $\left(c/2\sqrt{2}\right)f(\xi)+q(\xi)=1$.

4. Numerical result and discussion

Note that $\lim\limits_{X\to\infty} Q(X) = Q_\infty = 1$, (16)

but the solution has a singularity at $X = -\eta$, at which point both F and Q are infinite. This infinity may or may not have physical significance; we will discuss some possibilities. An interesting point is that if we restrict the range of X as in Figure 1, then for a suitable value of η the graph of the new solution has a certain similarity to the numerical solution displayed by Di Bartolo and Dorsey [cf. Ref.2]. We proceed to consider such a graph.

As in Figure 1, we restrict the values of X to the interval $-20 \le X \le 20$. We 'normalize' the solution (10) so that the value of $F(X)$ at $X = -20$ equals unity. We do this by choosing the constant η to be equal to $20+\sqrt{2}$; this choice, which is fairly representative, enables us to work out the various functions explicitly and examine their behaviour. This could be considered as a 'comprehension exercise' to get a 'feel' for the functions and, possibly, for the underlying physical situation. With the chosen value of η, the functions F and Q take the following form:

$$F(X) = \frac{\sqrt{2}}{\left(20+\sqrt{2}+X\right)}; \quad Q(X) = 1 - \frac{(c/2)}{\left(20+\sqrt{2}+X\right)} \quad (17)$$

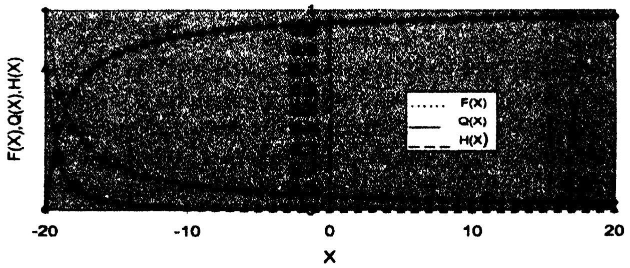

Figure 1. The graph of the solution of eq.(10) for the interval $-20 \le X \le 20$, for $\eta = 20+\sqrt{2}$, $c = 2\sqrt{2}$, displaying the functions F(X), Q(X) and H(X).

In this case, the solution (17) is finite and well defined for X in the chosen interval $-20 \le X \le 20$. Note that the magnetic field is given by

$$H = \partial_X q = Q'(X) = \frac{(c/2)}{\left(20+\sqrt{2}+X\right)^2}. \quad (18)$$

At $X = -20$, $H = c/4 = 1/\sqrt{2}$ (which is H_c in conventional units). Let us now choose two distinct values of c, namely: (i) $c = 2\sqrt{2}$ [in which case (13) reduces to (6)], and (ii) $c = \sqrt{2}$ the value chosen by Di Bartolo and Dorsey for their solution. This is for convenience and for some connection with the earlier work, which may or may not have physical significance. These choices do yield reasonable similarities with the numerical solution found by Di Bartolo and Dorsey. We discuss the two cases below.

Case -I: $c = 2\sqrt{2}$. In this case, eq.(13) reduces to (6), and we get

$$F(X) = \frac{\sqrt{2}}{\left(20+\sqrt{2}+X\right)}; \quad Q(X) = 1 - \frac{\sqrt{2}}{\left(20+\sqrt{2}+X\right)},$$

$$H(X) = \frac{\sqrt{2}}{\left(20+\sqrt{2}+X\right)^2} \quad (19)$$

$$F(-20) = 1,\ Q(-20) = 0,\ H(-20) = 1/\sqrt{2};$$

$$F(0) = \left(10\sqrt{2}+1\right)^{-1},\ Q(0) = 10\sqrt{2}/\left(10\sqrt{2}+1\right),$$
$$H(0) = 1/\sqrt{2}\left(10\sqrt{2}+1\right)^2; \quad (20)$$

$$F(20) = \left(20\sqrt{2}+1\right)^{-1},\ Q(20) = 20\sqrt{2}/\left(20\sqrt{2}+1\right),$$
$$H(20) = 1/\sqrt{2}\left(20\sqrt{2}+1\right)^2.$$

Case-II: $c = \sqrt{2}$. In this case, eq.(13) becomes $(1/2)F + Q = 1$, with

$$F(X) = \frac{\sqrt{2}}{\left(20+\sqrt{2}+X\right)}; \quad Q(X) = 1 - \frac{\sqrt{2}}{\sqrt{2}\left(20+\sqrt{2}+X\right)},$$

$$H(X) = Q'(X) = \frac{\ }{\sqrt{2}\left(20+\sqrt{2}+X\right)^2}. \quad (21)$$

$$F(-20) = 1,\ Q(-20) = 1/2,\ H(-20) = 1/2\sqrt{2};$$

$$F(0) = \left(10\sqrt{2}+1\right)^{-1},\ Q(0) = \left(20\sqrt{2}+1\right)/\left(20\sqrt{2}+1\right),$$
$$H(0) = 1/2\sqrt{2}\left(10\sqrt{2}+1\right)^2; \quad (22)$$

$$F(20) = \left(20\sqrt{2}+1\right)^{-1},\ Q(20) = \left(40\sqrt{2}+1\right)/\left(40\sqrt{2}+1\right),$$
$$H(20) = 1/2\left(20\sqrt{2}+1\right)^2.$$

The behaviour of the various curves in case (I) are depicted in Figure 1 and those of case (II) in Figure 2. In both cases, the curves representing $F(x)$ and $Q(X)$ are parts of rectangular hyperbolae. The various functions are worked out explicitly so that the reader can follow the results without numerical computation.

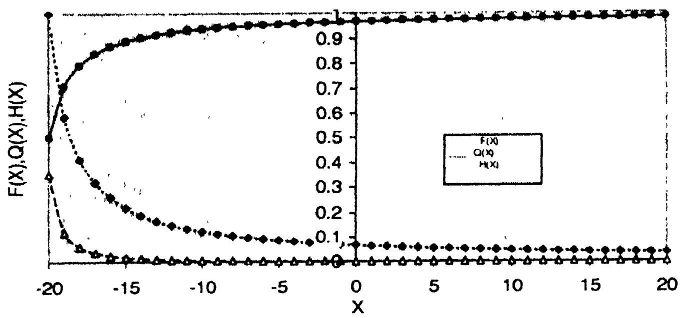

Figure 2. Same as in Figure 1, with $c = \sqrt{2}$

5. Concluding remarks

The problem of propagating fronts in superconductors is a general physical phenomenon, into which considerable understanding and insights have been gained in the past decade or two. The phenomenon may be termed as 'nonequilibrium growth patterns'; for example, the dendritic growth of solidifying systems (of which 'snowflakes' provide a good instance), or the fingured growth occurring at the interface of driven immiscible fluids [4,5]. The essential point is that in these systems, there is a competition between a dynamic instability, which causes the growth of a complex interface, and surface tension, which tends to smooth the interface. Here, apparently, anisotropy in the surface tension plays an important role [6,7]. The problem we have been considering in this work, namely, that related to 'the process of magnetic flux expulsion in type-I superconductors subjected [8,9] to a magnetic field quench', in the analogy with the solidification problem, the magnetic field may be compared with the temperature of the fluid, and the magnetic diffusion constant with the thermal diffusivity, and so on. We hope to consider these and related aspects in a future work, but here we mention only of these analogies, and point out that the form of the new solution [eqs.(10, 12)] which displays 'inverse length' rather than the 'exponential' decay of the earlier solution [eq.(8)], has behaviour which in a sense, may be complementary to the earlier solution and may be of physical interest. It may be possible to deal with the singularity, for example, by considering a situation in which it occurs outside the material of the superconductor.

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